Spring School in Nonlinear Partial Differential Equations
Brussels, May 17–21, 2010

Organizers: D. Bonheure (ULB), A. Ponce, J. Van Schaftingen (UCL), and C. Troestler (UMONS).

Advisory board: J.P. Gossez, E. Lami Dozo (ULB), J. Mawhin and M. Willem (UCL).

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1 Program

Rooms

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  - Forum H: CT 3B + CT 4B
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- Thready May 20:
  - Forum G
- Friday May 21:
  - Forum G

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	Mon	Tue	Wed	Thu	Fri
9:00	*Registration*	McKenna – L2	Plum – L2	Benci – L3	Cabré – L4
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12:00	PT – Serra	PT – Secchi	PT – Horák	Cabré – L3	McKenna – L4
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14:00	PT – Dupaigne	Benci – L2	Cabré – L2	Plum – L3	Plum – L4
14:30	PT – Dupaigne	Benci – L2	Cabré – L2	Plum – L3	
15:00	*Coffee break*	Benci – L2		Plum – L3	
15:30	CT – session 1	*Coffee break*		*Coffee break*	
16:00	CT – session 1	CT – session 3		McKenna – L3	
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2 Abstracts

2.1 Lectures

Variational methods in the study of solitary waves and hylomorphic solitons

Vieri Benci

These lectures are devoted to the study of solitary waves and solitons whose existence is related to the ratio energy/charge. These solitary waves are called hylomorphic. This class includes the Q-balls, which are spherically symmetric solutions of the nonlinear Klein-Gordon equation, as well as solitary waves and vortices which occur, by the same mechanism, in the nonlinear Schroedinger equation and in gauge theories. Mainly we will be interested in the very general principles which are at the base of the existence of solitons such as the Variational Principle, the Invariance Principle, the Noether's theorem, the Hamilton-Jacobi theory etc. We give a general definition of hylomorphic solitons and an interpretation of their nature (swarm interpretation) which is very helpful in understanding their behavior. We apply these ideas to the Nonlinear Schroedinger Equation, to the Nonlinear Klein-Gordon Equation and to the Klein-Gordon-Maxwell system.

Phase transitions and front propagation for fractional diffusion equations

Xavier Cabré

Long-range or "anomalous" diffusions, such as diffusions given by the fractional powers $(-\Delta)^s$ of the Laplacian, attract lately great interest in Physics, Biology, and Finance. They appear in diffusions in plasma, dislocations in crystals, in finance (American options modelled with jump processes), in geophysical fluid dynamics (the quasi-geostrophic equation), in certain reaction fronts and flames, and in population dynamics. The fractional powers of the Laplacian are the infinitesimal generators of the symmetric stable Lévy diffusion processes. These —also called Lévy flights— are diffusion processes that combine Brownian motion together with a jump process. From the mathematical point of view, nonlinear analysis for fractional diffusions has been mostly developed in the last years.

In these lectures, I will describe recent results concerning phase transitions for the fractional elliptic Allen-Cahn equation, as well as front propagation for the nonlinear fractional heat equation. We will present some recent results and open problems on the fractional analogue of the conjecture of De Giorgi on 1-d symmetry.

In a work in collaboration with Y. Sire, we are concerned with the equation

$$(-\Delta)^s u = f(u) \qquad \text{in } \mathbb{R}^n,$$

with $s \in (0,1)$. The case $s = 1/2$ was studied with J. Solà-Morales in [1]. Crucial to our analysis for $s \in (0,1)$ is a result of [2] which allows to realize this nonlocal equation as a degenerate elliptic equation posed in $\mathbb{R}^{n+1}_+$ together with a nonlinear Neumann boundary condition on $\mathbb{R}^n = \partial\mathbb{R}^{n+1}_+$. We characterize the nonlinearities f for which there exists a "layer" solution —meaning, essentially, a solution increasing in one direction. We establish several properties of these solutions, such as their uniqueness in $\mathbb{R}$, minimality, symmetry in certain dimensions, and decay at infinity. In collabortation with E. Cinti, we find sharp energy estimates for these and other solutions (such as "saddle-shaped" solutions). These estimates allow to improve the 1-d symmetry results of De Giorgi type for the nonlocal equation.

In collaboration with J.-M. Roquejoffre, we study the propagation of fronts for the fractional KPP equation

$$\partial_t u + (-\Delta)^s u = u(1-u) \quad \text{in } (0,\infty)\times\mathbb{R}^n, \qquad 0 \leqslant u \leqslant 1,$$

with $s \in (0,1)$. In [3], by heuristic considerations it was predicted that fronts should propagate at exponential speed —in contrast with the classical case $s = 1$ for which there is propagation at a constant KPP speed. In particular, no traveling wave should exist when $s < 1$. We establish mathematically these results. For instance, given an initial condition with compact support in $\mathbb{R}^n$, we prove that every level set of u is located at time t, up to an error, near $\{|x| = \exp(\mu^* t)\}$, where $\mu^* = f'(0)/(n + 2s)$ and $f(u)$ is equal to $u(1 - u)$ or to another concave monostable nonlinearity. Such exponential speed originates from the fact that the fundamental solution of the fractional heat equation has a power decaying tail at infinity — instead of the exponential tail of the Gaussian corresponding to $s = 1$.

Finally, I will describe results in collaboration with Jinggang Tan on the problem

$$\begin{cases} (-\Delta_{\mathrm{Dir}})^{1/2} u = u^p & \text{in} \quad \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in} \quad \Omega, \end{cases}$$

where $(-\Delta_{\mathrm{Dir}})^{1/2}$ stands for the unique positive square root of the Laplacian in a bounded domain $\Omega \subseteq \mathbb{R}^n$ with zero Dirichlet boundary conditions. Using also a local realization of the problem in the cylinder $\Omega \times [0, \infty)$, we establish existence and regularity results of positive solutions as well as a priori estimates of Gidas-Spruck type for subcritical powers, Liouville type theorems in a half space, and a symmetry result of Gidas-Ni-Nirenberg type.

References

[1] Cabré, X. and Solà-Morales, J., *Layer solutions in a half-space for boundary reactions*, Comm. Pure Appl. Math., **58**, 2005, 1678–1732.

[2] Caffarelli, L. and Silvestre, L., *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32**, 2007, 1245–1260.

[3] Mancinelli, R., Vergni, D., and Vulpiani, A., *Front propagation in reactive systems with anomalous diffusion*, Physica D. Nonlinear Phenomena, **185**, 2003, 175–195.

Semilinear boundary value problems: Nonlinear analysis, approximation and computer assistance

P. Joe McKenna

Lecture 1. We survey the main algorithms for finding approximate solutions for semilinear problems, including steepest descent, Newton's method, continuation methods and mountain pass algorithms.

Lecture 2. We discuss existence, multiplicity and uniqueness results for semilinear ellipitc problems, including computer assisted methods.

Lecture 3. We discuss a family of results for fourth order semilinear beam equations, again discussing numerical solutions and verification. This will include homoclinic and periodic solutions.

Lecture 4. We discuss periodic solutions of a semilinear spring equation and various methods to find them approximately.

Computer-assisted existence and multiplicity proofs for semilinear elliptic boundary value problems

Michael Plum

Many boundary value problems for semilinear elliptic partial differential equations allow very stable numerical computations of approximate solutions, but are still lacking analytical existence proofs. We propose a method which exploits the knowledge of a "good" numerical approximate solution, in order to provide a rigorous proof of existence of an exact solution close to the approximate one. This goal is achieved by a fixed-point argument which takes all numerical errors into account, and thus gives a mathematical proof which is not "worse" than any purely analytical one. The method is used to prove existence and multiplicity statements for some specific examples, including cases where purely analytical methods had not been successful.

2.2 Plenary talks

Exponentiation of a function and its Laplacian

Louis Dupaigne

The talk will discuss a priori L^∞ estimates, for solutions of the following Lane-Emden-like equation

$$(-\Delta)^s u = u^p \qquad \text{in } \Omega,$$

where $s = 1$ (Laplace operator), $s = 2$ (biharmonic operator) or $s \in (0,1)$ (fractional Laplacian). I will mostly focus on the supercritical case (p large) and discuss how regularity (and even partial regularity) is deeply related to the stability of the solution.

The deformation lemma and boundedness of Palais-Smale sequences

Jiří Horák

In the calculus of variations the well-known mountain pass theorem is an example of a minimax principle. These principles are based on the deformation lemma which shows how level sets of a functional are deformed in the descent direction (gradient flow). The functional needs to have a certain compactness property. In the talk a one-parameter family of functionals

$$I(\lambda, u) = I_0(u) - \lambda J(u)$$

admitting unbounded Palais-Smale sequences in a reflexive Banach space will be considered. A modified gradient flow will be constructed in order to prove a deformation lemma. This is a joint work with Marcello Lucia.

Recent results for the Schrödinger-Newton (or maybe Choquard) equation

Simone Secchi

In this talk I will review on some classical, recent, and new results for the non-local equation

$$-\hbar^2\Delta u + V(x)u = \frac{1}{\hbar^2}\left(\int_{\mathbb{R}^3} W(x-y)|u(y)|^2\,\mathrm{d}y\right)u \quad \text{in } \mathbb{R}^3.$$

This equation appears in several models: by coupling a Schrödinger equation to a Poisson equation (following an idea of Penrose), or directly as a variational problem (a particular case of the Choquard/Hartree equation, following e.g. P.-L. Lions). My results have been obtained in collaboration with S. Cingolani, M. Clapp, L. Jeanjean, M. Squassina.

On some supercritical Neumann problems

Enrico Serra

It is well known that the homogeneous Dirichlet problem for the equation $-\Delta u = u^{p-1}$ on a ball of $\mathbb{R}^N$ has no positive solutions if p is large (greater than or equal to the critical Sobolev exponent). The same happens, *mutatis mutandis*, for nonlinearities of the form $a(|x|)\,u^{p-1}$. On the contrary, the homogeneous Neumann problem $-\Delta u + u = u^{p-1}$ always admits the constant solution $u \equiv 1$, independently of the size of p. If one considers the nonlinearity $a(|x|)\,u^{p-1}$, the constant solution disappears, and the question of the solvability of the problem for p large becomes nontrivial. A major obstacle is the fact that one cannot work variationally, since the energy functional associated to the problem is not well defined in H^1, and not even in H^1_{rad}.

In this talk we will show that, under a qualitative assumption on a, the Neumann problem can still be treated variationally, and admits a nontrivial solution for every p. The method consists in the construction of a variational principle on a set of *monotone* functions, a device which does not seem to have been tried before.

2.3 Contributed talks

Kirchhoff systems with static and dynamic boundary damped conditions

Giuseppina Autuori

We treat the question of the non-existence of global solutions, or their long time behavior, of nonlinear hyperbolic Kirchhoff systems, under dynamic boundary damped conditions. The main p-Kirchhoff operator may be affected by a perturbation which behaves like $|u|^{p-2}u$ and the systems also involve an external force f and a nonlinear boundary damping Q. When $p = 2$, we consider some problems involving a higher order dissipation term. For them we give criteria in order to have $\|u(t,\cdot)\|_q \to \infty$ as $t \to \infty$ along any global solution $u = u(t,x)$, where q is a parameter related to the growth of f in u. Special subcases of f and Q are considered in applications.

Infinitely many positive solutions for a Schrödinger-Poisson system[1]

Pietro d'Avenia

We consider the nonlinear Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + V(|x|)\varphi u = |u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\Delta \varphi = V(|x|)u^2, & x \in \mathbb{R}^3, \end{cases} \qquad (\mathcal{SP})$$

where $p \in (1,5)$ and $V : \mathbb{R} \to \mathbb{R}$ is a nonnegative bounded function such that

$$V(r) = \frac{a}{r^m} + O\left(\frac{1}{r^{m+\theta}}\right), \quad \text{as } r \to +\infty.$$

Applying a method introduced by J. Wei and S. Yan in [1], we show that the problem $(\mathcal{SP})$ has infinitely many nonradial positive solutions (with a large number of bumps).

[1] Joint work with Alessio Pomponio and Giusi Vaira

References

[1] J. Wei, S. Yan, *Infinitely many positive solutions for the nonlinear Schrödinger equations in* $\mathbb{R}^N$, Calc. Var. Partial Differential Equations, **37** (2010), 423–439.

Global Nonexistence and Blow up for Nonlinear Polyharmonic Kirchhoff Systems

Francesca Colasuonno

We treat the question of the non continuation of local solutions for polyharmonic Kirchhoff systems

$$u_{tt} + M\left(\|u(t,\cdot)\|^2\right)(-\Delta)^L u + \mu u + Q(t,x,u,u_t) = f(t,x,u),$$

under homogeneous Dirichlet boundary conditions in bounded domains. Here $u = u(t,x)$ is the vectorial displacement, M the Kirchhoff function, Q a nonlinear damping, f an external source force and μ is a non–negative parameter. We also give an estimate of the blow–up time for local solutions.

Moreover we study the blow up at infinity for global solutions of strongly damped Kirchhoff systems involving two different (but related) Kirchhoff functions M and N, as well as lower order terms.

Concrete applications are considered in special subcases of f and Q.

Generic multiplicity for a scalar field equation

Francesca De Marchis

We consider the scalar field equation

$$-\Delta_g u + \rho = \rho \frac{h(x)\,e^u}{\int_\Sigma h(x)\,e^u\ dV_g}, \qquad x \in \Sigma,\ u \in H^1(\Sigma),$$

where (Σ, g) is a compact Riemannian surface, Δ_g is the Laplace-Beltrami operator on Σ, $h \in C^2(\Sigma)$ is a positive function and ρ is a real parameter.

The study of this problem has further motivations indeed this PDE arises in statistical mechanics as a mean field equation for the Euler flow and it also concerns the description of self-dual condensates of some Chern-Simon-Higgs model. Moreover this equation is related to the problem of prescribing the Gauss curvature of a surface via a conformal transformation of the metric.

Since the equation has a variational structure, solutions can be found as critical points of the Euler-Lagrange functional. Studying the topology (in particular the Betti numbers) of high and low sublevels of the functional and using Morse inequalities we obtain generic multiplicity of solutions for non-critical values of the parameter ρ, i.e. $\rho \notin 8\mathbb{N}\pi$.

In particular our result improves significantly the multiplicity estimate which can be deduced from the degree-counting formula proved by C.C. Chen and C.S. Lin. By this approach we are also able to derive generic existence and multiplicity of solutions for critical values of ρ.

A global bifurcation result for a semilinear elliptic equation

Francesca Gladiali

Let us consider the problem

$$\begin{cases} -\Delta u = u^p + \lambda u & \text{in } A \\ u > 0 & \text{in } A \\ u = 0 & \text{on } \partial A \end{cases} \tag{1}$$

where A is an annulus of $\mathbb{R}^N$, i.e. $A := \{x \in \mathbb{R}^N : a < |x| < b\}$, $b > a > 0$, $N \geqslant 2$, $p \in (1, +\infty)$ and $\lambda \in (-\infty, 0]$.

Recall that problem (1) has a radial solution for any $p \in (1, +\infty)$, and that this radial solution is unique if $\lambda \in (-\infty, 0]$.

We will denote by u_p this radial solution and by $\mathcal{S}$ the curve of radial solutions of (1) in the product space $(1, +\infty) \times C_0^{1,\alpha}(\overline{A})$. Let us recall that, given the curve $\mathcal{S}$ of radial solutions, a point

$(\bar{p}, u_{\bar{p}}) \in \mathcal{S}$ is a *nonradial bifurcation point* if in every neighbourhood of $(\bar{p}, u_{\bar{p}})$ in $(1, +\infty) \times C_0^{1,\alpha}(\overline{A})$ there exists a nonradial solution (p, v_p) of (1).

We will prove the existence of a sequence of values p_k such that $p_k \to +\infty$, and (p_k, u_{p_k}) is a nonradial bifurcation point. Moreover the bifurcation is global in the sense that from this values a continuum (i.e. a closed connected set) of nonradial solutions bifurcates. This continuum of solutions obeys at the so called Rabinowitz alternative, i.e. either it is unbounded in the space $(1, +\infty) \times C_0^{1,\alpha}(\overline{A})$ or it must meet some other bifurcation point (p_h, u_{p_h}).

Finally, in the case of low dimension, we were able to show that this continuum is unbounded. Some open problems are discussed.

These results are collected in [1, 2, 3].

References

[1] F. Gladiali, A global bifurcation result for a semilinear elliptic equation, accepted in J. of Mathematical Analysis and Applications.

[2] F. Gladiali, A global bifurcation result for a semilinear elliptic equation II, preprint.

[3] F. Gladiali, M. Grossi, F. Pacella and P.N. Srikanth, Bifurcation and symmetry breaking for a class of semilinear elliptic equations in an annulus, accepted in Calc. of Var.

On very weak positive solutions to some singular semilinear elliptic problems

Jesús Hernández

We use some recent work by Díaz-Rakotoson in order to obtain results concerning existence and regularity of very weak positive solutions to some singular semilinear elliptic problems. This extends and clarifies previous work.

Very weak solutions to elliptic systems with nonlinear boundary conditions

Ivana Kosírová

Consider the system

$$\left.\begin{aligned} -\Delta u &= h(.\,,v), \\ -\Delta v &= j(.\,,u), \end{aligned}\right\} \text{ in } \Omega,$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, complemented by the nonlinear Robin boundary condition on $\partial\Omega$:

$$\left.\begin{aligned} \partial_\nu u &= f(.\,,v) - u, \\ \partial_\nu v &= g(.\,,u) - v, \end{aligned}\right\} \text{ on } \partial\Omega.$$

We show that any very weak solution of this system belongs to $L^\infty(\Omega) \times L^\infty(\Omega)$ provided h, j, f, g satisfy suitable growth conditions. Our growth conditions are essentially optimal. We also consider more general problems and obtain a priori bounds of nonnegative solutions.

Elliptic problems with singular natural growth lower order terms

Pedro J. Martínez-Aparicio

We review some existence and nonexistence results [1, 5] for the quasilinear elliptic boundary value problem

$$\begin{cases} -\Delta u + g(u)|\nabla u|^2 = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open in $\mathbb{R}^N$, g is a continuous function with a possible singularity at zero, and f is a suitable integrable function. We describe the relation of our results with the papers [3, 4].

Moreover, we apply [2] the Bifurcation Theory to study the case of nonlinear data $f(x,u)$ in the right hand side of the problem.

References

[1] D. Arcoya, J. Carmona, T. Leonori, P.J. Martínez-Aparicio, L. Orsina and F. Petitta, *Existence and nonexistence of solutions for singular quadratic quasilinear equations*. J. Differential Equations **246** (2009), 4006–4042.

[2] D. Arcoya, J. Carmona and P.J. Martínez-Aparicio, *Bifurcation for quasilinear elliptic singular bvp*. Preprint 2009.

[3] L. Boccardo, *Dirichlet problems with singular and quadratic gradient lower order terms*, ESAIM Control Optim. Calc. Var., **14** (2008) 411-426.

[4] D. Giachetti and F. Murat, *An elliptic problem with a lower order term having singular behaviour.* Boll. Un. Mat. Ital. B, II (2009) 349–370.

[5] P.J. Martínez-Aparicio, *Singular Dirichlet problems with quadratic gradient.* Boll. Unione Mat. Ital. (9) II (2009) 559–574.

Rotationally symmetric 1-harmonic flows from D^2 to S^2

Salvador Moll

Given $\Omega \subseteq \mathbb{R}^N$, $p \geqslant 1$, a smoothly embedded compact submanifold M of $\mathbb{R}^{N+1}$ and $\mathbf{u} : \Omega \to M$, let

$$E_p(\mathbf{u}) := \frac{1}{p} \int_\Omega |\nabla \mathbf{u}|^p \,\mathrm{d}x \tag{2}$$

The p-harmonic flow for E_p is given by

$$\mathbf{u}_t = -\pi_{\mathbf{u}}\big(-\operatorname{div}(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u})\big), \tag{3}$$

where $\pi_{\mathbf{u}}$ denotes the orthogonal projection of $\mathbb{R}^{N+1}$ onto the tangent space $T_{\mathbf{u}}M$ of M at $\mathbf{u} \in M$. We consider the case in which $N = 2$, $\Omega = D^2 = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ x_1^2 + x_2^2 \leqslant 1\}$ and $M = S^2 = \big\{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_1^2 + x_2^2 + x_3^2 = 1\big\}$. In this case, (3) may be explicitly written as

$$\mathbf{u}_t = \operatorname{div}\big(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\big) + \mathbf{u}|\nabla \mathbf{u}|^p. \tag{4}$$

The Dirichlet problem for (4) amounts to impose the initial-boundary condition

$$\mathbf{u} = \mathbf{u}_0 \quad \text{on} \ \ \partial\big((0,\infty) \times \Omega\big). \tag{5}$$

For $p = 1$, (4) may be seen as a constrained gradient system of total variation:

$$\mathbf{u}_t = \operatorname{div}\left(\frac{D\mathbf{u}}{|D\mathbf{u}|}\right) + \mathbf{u}|D\mathbf{u}|. \tag{6}$$

In this talk I will make an overview of what is known for the problem (4)–(5)(specially for the $p = 2$ case) and I will present some new results obtained in collaboration with R. Dal Passo and L. Giacomelli for the special case of rotationally symmetric solutions to (6)–(5).

Saddle-point-type solutions of the bi-stable equation

Josef Otta

In this talk, we treat the quasilinear problem

$$\begin{cases} \varepsilon^p \Delta_p u(x) - W'(u(x)) = 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where Δ_p stands for the p-Laplacian, W is a double-well potential (a typical choice is $W(s) = (1-s^2)^2$) and Ω is a bounded domain. We show the existence of saddle-point type solutions by standard variational method employing unique positive solutions. Theoretical results are accompanied by numerical experiments which illustrate the solution dependence on the parameters p, ε and the shape of W in the terms of solution diagrams.

On the concentration of solutions of singularly perturbed biharmonic equations[2]

Marcos T.O. Pimenta

In this work, we study some questions about the existence and concentration of solutions to the following problem

$$\varepsilon^4 \Delta^2 u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N, \tag{7}$$

where $\varepsilon > 0$, $N \geqslant 5$ and the nonlinearity f is sublinear at the origin, superlinear and subcritical at infinity. The potential V is assumed to be nonegative and satisfies the following assumption:

$$\text{There is an open and bounded set } \Omega \subseteq \mathbb{R}^N \text{s.t. } \inf_{\partial\Omega} V < \inf_{\mathbb{R}^N} V.$$

The main result is:

THEOREM. *Under these assumptions, for all sequences $\varepsilon_n \to 0$ there is a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ such that the problem (7) has a solution u_{ε_k} in $H^2(\mathbb{R}^N)$. Furthermore, if x_{ε_k} is a maximum point of u_{ε_k}, then $\lim_{k\to\infty} V(x_{\varepsilon_k}) = \inf_{\mathbb{R}^N} V$.*

In order to prove this result we first modify the nonlinearity into one more convenient to apply the mountain pass theorem. The phenomenon of concentration is established combining an upper estimate of the energy of the solution obtained with the uniform decay of the solutions.

On the Schrödinger-Maxwell equations in presence of a general nonlinear term

Alessio Pomponio

In quantum electrodynamics, the interaction between a charge particle and the electromagnetic field is described by

[2]Supported by CNPq - Brazil

nonlinear Schrödinger-Maxwell equations which, in the electrostatic case, become

$$\begin{cases} -\Delta u + q\varphi u = g(u) & \text{in } \mathbb{R}^3, \\ -\Delta \varphi = q u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{$\mathcal{SM}$}$$

where $q > 0$.

In [1], we treat ($\mathcal{SM}$) assuming on the nonlinearity g the same general assumptions of Berestycki & Lions [3] proving the existence of a nontrivial positive solution for q sufficiently small. Moreover, using an abstract multiplicity result of [2], we show that, for any $n \in \mathbb{N}$, there exists $q_n > 0$ such that for any $0 < q < q_n$, ($\mathcal{SM}$) possesses at least n nontrivial solutions.

References

[1] A. Azzollini, P. d'Avenia, A. Pomponio, *On the Schrödinger-Maxwell equations under the effect of a general nonlinear term*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **27**, 779–791.

[2] A. Azzollini, P. d'Avenia, A. Pomponio, *Multiple critical points for a class of nonlinear functionals*, preprint.

[3] H. Berestycki, P.L. Lions, *Nonlinear scalar field equations. I. Existence of a ground state*, Arch. Rational Mech. Anal., **82**, (1983), 313–345.

Second order Moser type inequalities: a borderline case

Cristina Tarsi

We study optimal embeddings for the space of functions whose laplacian Δu belongs to $L^1(\Omega)$, where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain. Let $W^{2,p}(\Omega)$ denotes the second order Sobolev space, endowed with the standard norm $\|u\|_{2,p} = \sum_{\alpha \leqslant 2} \|D^\alpha u\|_p$. Then, provided $\partial\Omega$ is sufficiently smooth, the following continuous embedding holds

$$W^{2,p}(\Omega) \hookrightarrow L^{Np/(N-2p)}(\Omega) \qquad \text{if} \quad 1 \leqslant p < \frac{N}{2}.$$

In the limiting case $p = \frac{N}{2}$ we have a striking difference between the cases $p > 1$ and $p = 1$: indeed,

$$W^{2,N/2}(\Omega) \hookrightarrow L^{\varphi(u)}(\Omega) \qquad \text{if } N > 2, \quad \text{i.e. } p > 1,$$

where $L^{\varphi(u)}$ is the Orlicz space generated by the Young function $\varphi(u) = e^{|u|^{N/(N-2)}}$, while

$$W^{2,1}(\Omega) \hookrightarrow L^\infty(\Omega) \qquad \text{if } N = 2, \quad \text{i.e. } p = 1.$$

The situation changes if we consider the space of functions whose Laplacian belongs to $L^1(\Omega)$: we establish here a sharp embedding inequality into the Zygmund space $L_{\exp}(\Omega)$. From one side, this result enables us to improve the Brezis–Merle regularity estimate for the Dirichlet problem

$$\begin{cases} \Delta u = f(x), & \Omega \\ u = 0, & \partial\Omega \end{cases}$$

where $f \in L^1(\Omega)$; on the other side, it represents a borderline case of D.R. Adams generalizations of Trudinger-Moser type inequalities to the case of higher order derivatives. The results discussed in this talk have been obtained in collaboration with B. Ruf and D. Cassani (University of Milan) and appeared in [Ann. I. H. Poincaré, 27 (2010), 73-93].

The concentration-compactness principle for variable exponent spaces and applications

Analia Silva

In this work join with Julián Fernández, we extend the well-known concentration-compactness principle of P.-L. Lions to the variable exponent case. More precisely, we prove: Let $q(x)$ and $p(x)$ be continuous functions such that $q(x) \leqslant p(x)$ in Ω. Let $\{u_j\}$ be a weakly convergent sequence in $W_0^{1,p(x)}(\Omega)$ with weak limit u, and such that:

- $|\nabla u_j|^{p(x)} \rightharpoonup \mu$ weakly-$*$ in the sense of measures;
- $|u_j|^{q(x)} \to \nu$ weakly-$*$ in the sense of measures.

Assume, moreover that $\mathcal{A} = \{x \in \Omega \colon q(x) = p^*(x)\}$ is nonempty. Then, for some countable index set I we have:

$$\nu = |u|^{q(x)} + \sum_{i\in I} \nu_i \delta_{x_i} \quad \nu_i > 0 \tag{8}$$

$$\mu \geqslant |\nabla u|^{p(x)} + \sum_{i\in I} \mu_i \delta_{x_i} \quad \mu_i > 0 \tag{9}$$

$$S\nu_i^{1/p^*(x_i)} \leqslant \mu_i^{1/p(x_i)} \qquad \forall i \in I. \tag{10}$$

where $\{x_i\}_{i\in I} \subseteq \mathcal{A}$ and S is the best constant in the Gagliardo-Nirenberg-Sobolev inequality for variable exponents, namely

$$S := \inf_{\varphi \in C_0^\infty(\mathbb{R}^n)} \frac{\| |\nabla \varphi| \|_{L^{p(x)}(\mathbb{R}^n)}}{\|\varphi\|_{L^{p^*(x)}(\mathbb{R}^n)}}.$$

We want to remark that in this Theorem is not required the exponent $q(x)$ to be critical *everywhere* and that the point masses are located in the *criticality set* $\mathcal{A} = \{x \in \Omega \colon q(x) = p^*(x)\}$. Moreover as an application of this Theorem , following the techniques of [1] we prove the existence of solutions for problems with lack of compactness.

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Positive solutions for some non autonomous Schrödinger-Poisson Systems

Giusi Vaira

In a joint work with Prof. G. Cerami, we study the Schrödinger-Poisson system:

$$\begin{cases} -\Delta u + u + K(x)\varphi(x)u = a(x)|u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\Delta\varphi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{SP}$$

with $p \in (3,5)$. Assuming that $a : \mathbb{R}^3 \to \mathbb{R}$ and $K : \mathbb{R}^3 \to \mathbb{R}$ are nonnegative functions such that $\lim_{|x|\to\infty} a(x) = a_\infty > 0$, $\lim_{|x|\to\infty} K(x) = 0$ and are satisfying suitable assumptions, but not requiring any symmetry property on them, we prove the existence of positive solutions. We also discuss recent developments in a preprint paper, assuming that $\lim_{|x|\to\infty} a(x) = a_\infty > 0$, and $\lim_{|x|\to\infty} K(x) = K_\infty > 0$. An analysis far from trivial allows us to find positive solutions of (SP).

On a conjecture of Brezis and Vázquez about the extremal solution

Salvador Villegas

Consider the following semilinear elliptic equation, which has been extensively studied:

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega\,, \\ u \geqslant 0 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \qquad (P_\lambda)$$

where $\Omega \subseteq \mathbb{R}^N$ is a smooth bounded domain, $N \geqslant 1$, $\lambda \geqslant 0$ is a real parameter, and the nonlinearity $f : [0,\infty) \to \mathbb{R}$ satisfies

$$f \text{ is } C^1, \text{ nondecreasing and convex, } f(0) > 0, \text{ and} \\ \lim_{u\to+\infty} \frac{f(u)}{u} = +\infty. \qquad (11)$$

It is well known that there exists a finite positive extremal parameter λ^* such that (P_λ) has a minimal classical solution $u_\lambda \in C^2(\overline{\Omega})$ if $0 \leqslant \lambda < \lambda^*$, while no solution exists, even in the weak sense, for $\lambda > \lambda^*$. The set $\{u_\lambda : 0 \leqslant \lambda < \lambda^*\}$ forms a branch of classical solutions increasing in λ. Its increasing pointwise limit $u^*(x) := \lim_{\lambda\uparrow\lambda^*} u_\lambda(x)$ is a weak solution of (P_λ) for $\lambda = \lambda^*$, which is called the extremal solution of (P_λ) (see [1]).

The regularity and properties of the extremal solutions depend strongly on the dimension N, domain Ω and nonlinearity f.

If $\Omega = B_1$ (the unit ball of $\mathbb{R}^N$), it is easily seen that u^* is radial. In this framework, Cabré and Capella [2] have proved that $u^* \in L^\infty(\Omega)$ if $N \leqslant 9$. The result is sharp: if $N \geqslant 10$ and $f(u) = e^u$, then $u^*(r) = -2 \log r$ (see [3]). Moreover, in [4] sharp estimates of the extremal solution are obtained.

On the other hand, Brezis and Vázquez raised the following open problem (see [1, Problem 5]): which is the behavior of $f'(u^*)$ near the the singularities of u^*. Is it always like C/r^2 ? In the case $\Omega = B_1$, we will answer negatively to this question. In fact we will see that for every $N > 10$ and decreasing function $\Psi \in C(0, 1]$ verifying

$$\frac{2(N-2)}{r^2} \leqslant \Psi(r) \leqslant \frac{(N-2)^2}{4\, r^2}$$

there exists f satisfying (11), such that $\lambda^* = 1$ and $f'(u^*(r)) = \Psi(r)$, $0 < r \leqslant 1$.

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3 Participants

Colored names are links to the abstract if the person gives a talk.

AL KALALA Alphonse, Université Libre de Bruxelles (Belgium), ⟨Alphonse.Kalala.Beya@ulb.ac.be⟩.

AUTUORI Giuseppina, University of Perugia (Italy), ⟨autuori@dmi.unipg.it⟩.

BARNAŚ Sylwia, Jagiellonian University, Cracow (Poland), ⟨sylwia.barnas@im.uj.edu.pl⟩.

BENCI Vieri, Dipartimento di Matematica Applicata “Ulisse Dini”, Università di Pisa (Italy), ⟨benci@dma.unipi.it⟩.

BONHEURE Denis, Université Libre de Bruxelles (Belgium), ⟨denis.bonheure@ulb.ac.be⟩.

BOUGUIMA Sidi Mohammed, University of Tlemcen (Algeria), ⟨bouguima@yahoo.fr⟩.

BOUMAZA Nouri, University of Tebessa (Algeria), ⟨nboumaza@gmail.com⟩.

CABRÉ Xavier, ICREA and Univ. Politècnica de Catalunya (Spain), ⟨xavier.cabreupc.edu⟩.

CHOUDHARY Neelu, Barkatullah University, Bhopal (India), ⟨neelaakash2007@yahoo.co.in⟩.

COELHO Isabel, Université Libre de Bruxelles (ULB) (Belgium), ⟨Isabel.Coelho@ulb.ac.be⟩.

COLASUONNO Francesca, Università degli Studi di Bari (Italy), ⟨francesca.colasuonno@dmi.unipg.it⟩.

D'AVENIA Pietro, Politecnico di Bari (Italy), ⟨p.davenia@poliba.it⟩.

DE COSTER Colette, Université de Valenciennes (France), ⟨Colette.DeCoster@univ-valenciennes.fr⟩.

DE MARCHIS Francesca, SISSA (Italy), ⟨demarchis@sissa.it⟩.

DERLET Ann, Université Libre de Bruxelles (Belgium), ⟨ann.derlet@ulb.ac.be⟩.

Desmons Bertrand, Université de Mons (Belgium), ⟨Bertrand.Desmons@umons.ac.be⟩.

De Valeriola Sébastien, Université Catholique de Louvain (Belgium), ⟨DeValeriola@math.ucl.ac.be⟩.

Di Cosmo Jonathan, Université catholique de Louvain (Belgium), ⟨jonathan.dicosmo@uclouvain.be⟩.

Dipierro Serena, SISSA Trieste (Italy), ⟨dipierro@sissa.it⟩.

Dupaigne Louis, Université de Picardie (France), ⟨louis.dupaigne@math.cnrs.fr⟩.

Freisinger Hans-Jürgen, Karlsruhe Institute of Technology (Germany), ⟨hans-juergen.freisinger@student.kit.edu⟩.

Gladiali Francesca, Dipartimento di Matematica, Università di Sassari (Italy), ⟨fgladiali@uniss.it⟩.

Goncerz Paweł, Jagiellonian University, Cracow (Poland), ⟨pawel.goncerz@im.uj.edu.pl⟩.

Gossez Jean-Pierre, Université Libre de Bruxelles (Belgium), ⟨gossez@ulb.ac.be⟩.

Grumiau Christopher, Université de Mons (Belgium), ⟨christopher.grumiau@umons.ac.be⟩.

Hernández Jesús, Universidad Autonoma de Madrid (Spain), ⟨jesus.hernandez@uam.es⟩.

Horák Jiří, Department of Mathematics, University of Cologne (Germany), ⟨jhorak@math.uni-koeln.de⟩.

Jhangeer Adil, Lahore university of management sciences (Pakistan), ⟨adilj@lums.edu.pk⟩.

Kamel Haouam, Université de Tebessa (Algeria), ⟨haouam@yahoo.fr⟩.

Kosírová Ivana, Comenius University (Slovakia), ⟨ivana.kosirova@gmail.com⟩.

Kudreyko Aleksey, University of Salerno (Italy), ⟨akudreyko@unisa.it⟩.

Lami Dozo Enrique, Université Libre de Bruxelles (Belgium), ⟨lamidozo@ulb.ac.be⟩.

Łukasiak Renata, Nicolaus Copernicus University (Poland), ⟨renatka@mat.umk.pl⟩.

Malik Salman Amin, Université de La Rochelle (France), ⟨salman.malik@univ-lr.fr⟩.

Martínez-Aparicio Pedro J., Universidad de Granada (Spain), ⟨pedrojma@ugr.es⟩.

Mawhin Jean, Université Catholique de Louvain (Belgium), ⟨Jean.Mawhin@uclouvain.be⟩.

McKenna P. Joe, Department of Mathematics, University of Connecticut (USA), ⟨mckenna@math.uconn.edu⟩.

Mohamed Berbiche, Khenchela University (Algeria), ⟨berbichemed@yahoo.fr⟩.

Mohammadi Abbasali, University of Tabriz (Iran), ⟨alimohammadi@tabrizu.ac.ir⟩.

Moll Salvador, Universitat de València (Spain), ⟨j.salvador.moll@uv.es⟩.

Nasri Yasmina, Université de Tlemcen (Algeria), ⟨y_nasri@hotmail.com⟩.

Nguyen Loc, University of Utah (USA), ⟨loc@math.utah.edu⟩.

Oliveira Pimenta Marcos Tadeu, ICMC - Universidade de São Paulo (Brazil), ⟨pimenta@icmc.usp.br⟩.

Otta Josef, Department of Mathematics, University of West Bohemia (Czech Republic), ⟨jotta@kma.zcu.cz⟩.

Parini Enea, University of Cologne (Germany), ⟨eparini@math.uni-koeln.de⟩.

Plum Michael, Institut für Analysis, Universität Karlsruhe (Germany), ⟨michael.plum@math.uni-karlsruhe.de⟩.

Pomponio Alessio, Politecnico di Bari (Italy), ⟨a.pomponio@poliba.it⟩.

Ponce Augusto, Université Catholique de Louvain (Belgium), ⟨augusto.ponce@uclouvain.be⟩.

Presoto Adilson Eduardo, Université Catholique de Louvain (Belgium), ⟨adilsonpresoto@yahoo.com.br⟩.

Rihani Samira, MIA, université de La Rochelle (France), ⟨srihani@univ-lr.fr⟩.

Rykaczewski Krzysztof, Nicolaus Copernicus University (Poland), ⟨mozgun@mat.umk.pl⟩.

Said-Houari Belkacem, Savoie University (France), ⟨saidhouarib@yahoo.fr⟩.

Saoudi Khaled, University Khenchela (Algeria), ⟨saoudikhaled@hotmail.com⟩.

Secchi Simone, Dipartimento di Matematica e Applicazioni, Università di Milano "Bicocca" (Italy), ⟨Simone.Secchi@unimib.it⟩.

Serra Enrico, Politecnico di Torino (Italy), ⟨enrico.serra@polito.it⟩.

Shanidze Zauri, London (UK), ⟨zshanidze@hotmail.co.uk⟩.

Shioji Naoki, Yokohama National University (Japan), ⟨shioji@math.sci.ynu.ac.jp⟩.

Silva Analia, UBA (Argentina), ⟨asilva@dm.uba.ar⟩.

Tarsi Cristina, Dept. Mathematics, University of Milan (Italy), ⟨Cristina.Tarsi@unimi.it⟩.

Tran Ba Thuan, Université Catholique de Louvain (Belgium), ⟨thuan.tranba@uclouvain.be⟩.

Troestler Christophe, Université de Mons (Belgium), ⟨Christophe.Troestler@umons.ac.be⟩.

Vaira Giusi, SISSA (Italy), ⟨vaira@sissa.it⟩.

Van Schaftingen Jean, Université catholique de Louvain (Belgium), ⟨Jean.VanSchaftingen@uclouvain.be⟩.

Villegas Salvador, Universidad de Granada (Spain), ⟨svillega@ugr.es⟩.

Vrabel Vladimir, Ghent University (Belgium), ⟨vv@cage.ugent.be⟩.

Willem Michel, Université Catholique de Louvain (Belgium), ⟨Michel.Willem@uclouvain.be⟩.

4 Practical information

4.1 Lunches

A complimentary sandwich buffet will be offered every day in our cafeteria (see FORUM on the map of the campus next page).

4.2 Dinners

For dinners, you will find plenty of restaurants around the *Grand Place*. Brussels offers an international and eclectic choice of restaurants. More typical but rather touristic restaurants can be found in the *Rue des bouchers* and *Petite rue des bouchers* close to the Grand Place. Good alternatives are the *Quartier St Géry* and the *Place St Catherine* at walking distance from the *Grand Place*. Restaurants at the *Place St Catherine* are specialized in seafood and fish. Prices may vary a lot.

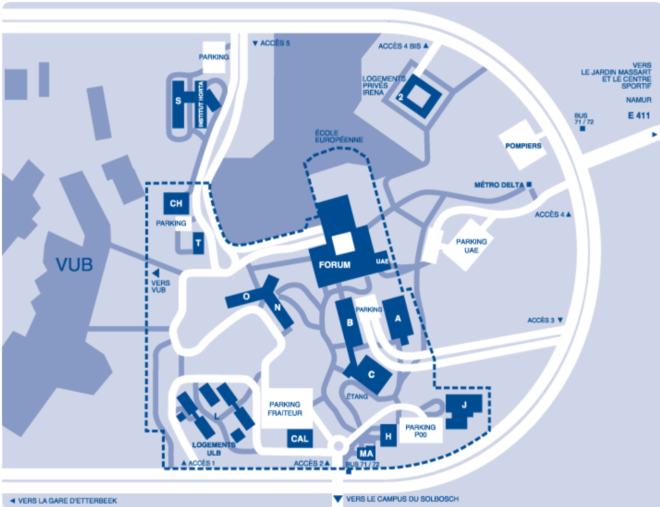
For quicker and more economical deals, you may try the kebabs, pasta or pizza in the *Rue Marché aux fromages* close to the *Grand Place*.

4.3 WiFi

Please ask the organizers for a login and password. The network is called “Plaine-WiFi”. You have to start your session in a browser. The rooms NO4.07 and NO4.006 (4th floor of building NO) are equipped with a good signal for WiFi users.

You may also be able to use “URBIZONE” which is a free network sponsored by the city of Brussels. After associating with the access point, you need to launch a browser and follow the instructions to create an account.

4.4 Map of the campus



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